Trigonometric Identities: Pythagorean Connection ACTIVITY 33

More Than Just Triangles

Lesson 33-1 The Pythagorean Identity

Learning Targets:

- Prove the Pythagorean identity.
- Use the Pythagorean identity to find sin θ, cos θ, or tan θ, given the value of one of these functions and the quadrant of θ.

SUGGESTED LEARNING STRATEGIES: Close Reading, Look for a Pattern, Discussion Groups, Create Representations

The trigonometric functions of sine, cosine, and tangent are each a ratio relating two of the three sides of a right triangle. Any two of these trigonometric ratios have one side in common, and together they relate all three sides of a triangle.

We can use the definitions of sine, cosine, and tangent to explore these relationships.

Look at the ratios that were defined in the previous lesson for the unit circle, where the length of the hypotenuse is equal to 1:

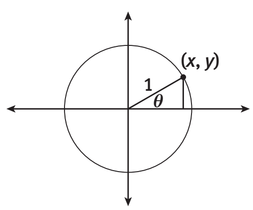
$\sin\theta = \frac{y}{1}$ $\qquad \cos\theta = \frac{x}{1}$ $\qquad \tan\theta = \frac{y}{x}$

Since $\sin\theta = y$ and $\cos\theta = x$, we can write the $\tan\theta$ in terms of sine and cosine.

$$\tan\theta = \frac{y}{x}, \text{ so } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

In geometry, you studied a special relationship between the sides of a right triangle when you learned the Pythagorean Theorem.

Let's express the relationship between the sides of a triangle on the unit circle with the Pythagorean Theorem.



Here we can see that the legs are x and y and the hypotenuse is 1, so $x^2 + y^2 = 1^2$. Simplified, $x^2 + y^2 = 1$.

My Notes

MATH TIP

The Pythagorean Theorem shows the following relationship between the sides of a right triangle: $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse of a right triangle.

Common Core State Standards for Activity 33

HSF-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin (θ), cos (θ), or tan (θ), given sin (θ), cos (θ), or tan (θ), and the quadrant of the angle.

ACTIVITY 33

Directed

Activity Standards Focus

In this activity, students will use the Pythagorean Theorem to derive the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$. They will then combine this identity with the reciprocal identities to derive related Pythagorean identities. Emphasis should be on identifying relationships, not memorization. Monitor students' progress to ensure that they can justify each step used as they derive identities and solve problems.

Lesson 33-1

PLAN

Pacing: 1 class period

Chunking the Lesson

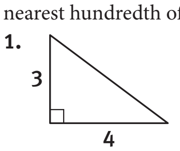
Example A
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to use the Pythagorean Theorem to find the missing side to the nearest hundredth of each triangle.

1.



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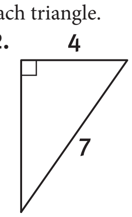

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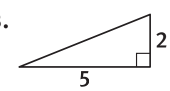

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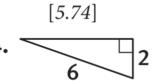

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Introduction Shared Reading, Marking the Text, Look for a Pattern, Create Representations Invite students to describe how they solved each problem in the Bell-Ringer Activity. Explain that they will use the Pythagorean Theorem in a new way during this activity. Either you or a student can read the introduction aloud as the class follows along and marks important information. Encourage students to draw pictures to illustrate the concepts discussed. This page contains a lot of important information, so take time to ensure that students understand the concepts presented.

Developing Math Language

Require proper use of the term *trigonometric identity* throughout this activity. Students should understand that an *identity* is an algebraic equation that is true for all values of the variable.

Teacher to Teacher

As students continue to study trigonometry and precalculus, their ability to verify trigonometric identities will prove to be an important skill. Allow students to struggle with each task—providing guidance too quickly may prevent students from making important connections on their own. It is not productive, however, for students to become frustrated. Provide guidance as needed so that this first experience with trigonometric identities is a positive one.

Example A Create Representations, Look for a Pattern, Discussion Groups, Debriefing Encourage students to support each other as they discover how to complete these exercises. Students should draw sketches to support their work. Monitor students' progress to ensure that they are creating accurate representations. Point out the Math Tip and have students explain how to connect the tip to the example.

Technology Tip

Remind students that precise answers are expected for each question in this activity. Students using calculators should set them to return exact values rather than decimal approximations. On the TI-83 or 84, after performing the calculation, press the MATH key. Because the first option is the fraction format, select ENTER or press 1. Press ENTER again to view the answer as a fraction.

For additional technology resources, visit SpringBoard Digital.

Differentiating Instruction

Invite individuals or small groups to present their work to the class. Encourage students to ask questions that will guide the presenters to explain their reasoning. Students should be able to explain which pictures, equations, and relationships are most helpful to them as they solve each problem.

My Notes

We can rewrite this equation with sine and cosine by substituting $\sin\theta$ for y and $\cos\theta$ for x. Now we have the following equation:

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Using the notation $\sin^2\theta$ for $(\sin\theta)^2$ and $\cos^2\theta$ for $(\cos\theta)^2$, this equation can be rewritten as follows:

$$\sin^2\theta + \cos^2\theta = 1$$

This relationship is called a Pythagorean identity.

We can use all of these relationships between sine, cosine, and tangent to solve problems on the unit circle.

MATH TIP

The sign of each trigonometric function depends on the quadrant in which the terminal side of the angle lies.

Example A

Given that $\cos\theta = -\frac{3}{5}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\sin\theta$ and $\tan\theta$.

Since we need $\sin\theta$ to calculate $\tan\theta$, let's first find $\sin\theta$.

Using $\sin^2\theta + \cos^2\theta = 1$, substitute any given information and solve.

$$\sin^2\theta + \left(-\frac{3}{5}\right)^2 = 1$$
$$\sin^2\theta + \left(\frac{9}{25}\right) = 1$$
$$\sin^2\theta = \frac{16}{25}$$
$$\sqrt{\sin^2\theta} = \sqrt{\frac{16}{25}}$$
$$\sin\theta = \frac{4}{5}$$

Because it is in the second quadrant, sine is positive.

Now we can find $\tan\theta$ using $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$\tan\theta = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

Try These A

a. Given that $\cos\theta = -\frac{8}{17}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\sin\theta$ and $\tan\theta$.

$\sin\theta = \frac{15}{17}$, $\tan\theta = -\frac{15}{8}$

b. Given that $\cos\theta = -\frac{5}{13}$ and that $\pi < \theta < \frac{3\pi}{2}$, find the value of $\sin\theta$ and $\tan\theta$.

$\sin\theta = -\frac{12}{13}$, $\tan\theta = \frac{12}{5}$

Lesson 33-1
The Pythagorean Identity

Check Your Understanding

1. Given that $\cos\theta = \frac{7}{25}$ and that $0 < \theta < \frac{\pi}{2}$, find the value of $\sin\theta$ and $\tan\theta$.
2. Given that $\sin\theta = \frac{3}{5}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\cos\theta$ and $\tan\theta$.

LESSON 33-1 PRACTICE

3. Given that $\sin\theta = -\frac{40}{41}$ and that $\pi < \theta < \frac{3\pi}{2}$, find the value of $\cos\theta$ and $\tan\theta$.
4. Given that $\cos\theta = -\frac{5\sqrt{3}}{10}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\sin\theta$ and $\tan\theta$.
5. Given that $\sin\theta = -\frac{4\sqrt{2}}{8}$ and that $\frac{3\pi}{2} < \theta < 2\pi$, find the value of $\cos\theta$ and $\tan\theta$.
6. **Reason quantitatively.** If sine and cosine are both positive, in which quadrant is the terminal side of the angle?
7. **Reason abstractly.** When solving for a missing value of sine or cosine in the equation $\sin^2\theta + \cos^2\theta = 1$, is it possible that the answer may be negative? Explain.

My Notes

ELL Support

Monitor group discussions to ensure that all students are participating and that discussions add to comprehension and understanding of how to apply math concepts. Encourage students to take notes during their group discussions to aid in comprehension and to enhance listening skills.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand how to use each piece of information presented. Encourage students to draw sketches to support their work. An accurate sketch will enable students to identify the correct absolute value and the correct sign for each trigonometric function.

Answers

1. $\sin\theta = \frac{24}{25}$, $\tan\theta = \frac{24}{7}$
2. $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 33-1 PRACTICE

3. $\cos\theta = -\frac{9}{41}$, $\tan\theta = \frac{40}{9}$
4. $\sin\theta = \frac{1}{2}$, $\tan\theta = -\frac{\sqrt{3}}{3}$
5. $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = -1$
6. Quadrant I
7. A square root can be positive or negative. Depending on which quadrant the terminal side of the angle lies in, sine and/or cosine may be negative.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand the relationships among the three primary trigonometric functions. Encourage struggling students to return to the right triangle relationships that they used when first defining trigonometric values. Draw a sketch of a right triangle and use the ratio relationships to label the lengths of the sides based on the information given. For example, if $\sin\theta = \frac{4}{5}$, then the leg opposite the angle θ has length 4 units, and the hypotenuse is 5 units long. The Pythagorean Theorem will supply the length of the missing side. Students can then use the ratios to identify the values of the other trigonometric functions. Demonstrate how students can change the orientation of their right triangle sketches to help them identify the correct sign of each trigonometric value.

Lesson 33-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 Example A
Check Your Understanding
Example B
Check Your Understanding
Lesson Practice

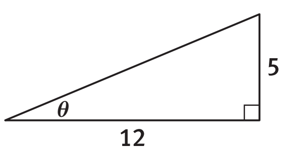
TEACH

Bell-Ringer Activity

Present the equation $x^2 + 2y = 4(y - x)$ to the class. Have students make the following substitutions and then simplify the resulting equations.

1. $x = 4a, y = 2b$ $[16a^2 + 16a - 4b = 0]$
2. $x = a^2, y = b^2$ $[a^4 + 4a^2 - 2b^2 = 0]$
3. $x = 3 - r, y = t + 2$ $[r^2 - 10r - 2t = -17]$

Introduction Discussion Groups, Close Reading, Marking the Text Monitor students' work as they identify and work with the reciprocal and quotient identities. Present students with a right triangle, like the one shown.



Challenge students to identify the values of all six trigonometric functions for θ. Supply additional triangles as needed until students have mastered the skill.

1 Think Aloud Invite students to explain their reasoning aloud as they establish the quotient identities.

Example A Create Representations, Look for a Pattern, Think-Pair-Share All three Pythagorean identities are important in calculus. While most students are successful at memorizing and applying the primary identity, $\sin^2\theta + \cos^2\theta = 1$, far fewer are able to remember the other two related identities. The work in this Example demonstrates that students do not need to memorize each equation. Encourage students to identify the relationships among the three Pythagorean identities.

My Notes

Learning Targets:

- Define the three reciprocal trigonometric functions.
- Use the Pythagorean identity and the reciprocal trigonometric functions to prove other trigonometric identities.

SUGGESTED LEARNING STRATEGIES: Think Aloud, Discussion Groups, Note Taking

In addition to sine, cosine, and tangent, there are three more trigonometric functions. These functions are secant (sec), cosecant (csc) and cotangent (cot). Each of these is a reciprocal of one of the first three trigonometric functions you have learned. Similarly, the first three can be considered reciprocals of the second three. The reciprocal identities are shown here.

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$
$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

1. In Lesson 33-1 you learned the tangent quotient identity, $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Given that cotangent and tangent are reciprocals of one another, express $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$.

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

In Lesson 33-1 you also learned about one Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$. There are three Pythagorean identities altogether. You can use the reciprocal and quotient identities to find the other two.

Example A

Divide $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ to find the second Pythagorean identity.

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Simplify each ratio and substitute single trigonometric functions.

$$\tan^2\theta + 1 = \sec^2\theta$$

This is the second Pythagorean identity.

In a similar way, you can find the third Pythagorean identity.

Try These A

Divide $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$ to find the third Pythagorean identity.

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

Lesson 33-2
Other Trigonometric Identities

ACTIVITY 33
continued

My Notes

Check Your Understanding

2. Simplify $\frac{1}{\cos\theta}$.

3. Show how $\sec^2\theta = \frac{1}{\cos^2\theta}$ is equivalent to $\sec\theta = \frac{1}{\cos\theta}$.

In addition to these trigonometric identities, there are other forms of the identities that you can derive by multiplying one identity by a trigonometric function.

Example B

Multiply $1 + \cot^2\theta = \csc^2\theta$ by $\sin\theta$ to find another form of the trigonometric identity. Remember that if you multiply both sides of an equation by the same expression, it does not change the equation.

$1 + \cot^2\theta = \csc^2\theta$

$\sin\theta(1 + \cot^2\theta) = \sin\theta(\csc^2\theta)$

$\sin\theta + \sin\theta(\cot^2\theta) = \sin\theta\left(\frac{1}{\sin^2\theta}\right)$ Write parts you can in terms of sine.

$\sin\theta + \sin\theta\cot^2\theta = \csc\theta$ Simplify.

Other forms of trigonometric identities can be found this way as well.

Try These B

Make use of structure. Multiply $\tan^2\theta + 1 = \sec^2\theta$ by $\cos\theta$ to find another form of the trigonometric identity.

$\tan^2\theta + 1 = \sec^2\theta$

$\cos\theta(\tan^2\theta + 1) = \cos\theta(\sec^2\theta)$

$\cos\theta\tan^2\theta + \cos\theta = \cos\theta\left(\frac{1}{\cos^2\theta}\right)$

$\cos\theta\tan^2\theta + \cos\theta = \sec\theta$

ACTIVITY 33 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand how to apply the reciprocal identities. Take time to discuss each question.

Answers

2. $\sec\theta$

3. $\sec\theta(\sec\theta) = \frac{1}{\cos\theta}\left(\frac{1}{\cos\theta}\right)$,

$\sec^2\theta = \frac{1}{\cos^2\theta}$

Teacher to Teacher

Throughout this activity, it is important to remember that students are building experience with trigonometric identities. If a student does not discover the way to verify an identity on his or her own, encourage him or her to make a note of the technique developed when working as a class. Students are building a "tool box" of ideas that will help them work with more challenging identities in the future. As long as students are adding techniques to their toolbox, then they are completing the activity successfully.

Example B Close Reading, Discussion Groups, Think-Pair-Share, Debriefing Encourage students to share their strategies with each other. Challenge students to apply this strategy in a different way to identify additional trigonometric identities.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand how to work with trigonometric identities. Require precise notation as students record their work.

Answers

4. $\sin\theta + \cos\theta\cot\theta = \csc\theta$
5. $\sin\theta\tan\theta + \cos\theta = \sec\theta$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 33-2 PRACTICE

5. $\sin^2\theta$
6. $\cot\theta$; $\dfrac{\cos\theta}{\sin\theta}$
7. $\sin^2\theta + \cos^2\theta = 1$
8. $\sin^2\theta\tan^2\theta + \sin^2\theta = \tan^2\theta$
9. No; the reciprocal identity of $\cos\theta$ is $\sec\theta$. Multiplying by $\cos\theta$ is the same as dividing by $\sec\theta$.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand basic concepts related to trigonometric identities. Some students may be overwhelmed when working with the newer trigonometric functions introduced in this activity. Remind students that the reciprocal and ratio identities can be used to rewrite any trigonometric equation in terms of sine and cosine alone. For many students, this is a valuable first step.

My Notes

Check Your Understanding

4. Multiply $\sin^2\theta + \cos^2\theta = 1$ by $\csc\theta$ to find another form of the trigonometric identity.

5. Divide $\sin^2\theta + \cos^2\theta = 1$ by $\cos\theta$ to find another form of the trigonometric identity.

LESSON 33-2 PRACTICE

5. Simplify $\dfrac{1}{\csc^2\theta}$.

6. Make use of structure. Write $\dfrac{1}{\tan\theta}$ in two other ways.

7. Multiply $1 + \cot^2\theta = \csc^2\theta$ by $\sin^2\theta$ to find another form of the trigonometric identity.

8. Multiply $\tan^2\theta + 1 = \sec^2\theta$ by $\sin^2\theta$ to find another form of the trigonometric identity.

9. Critique the reasoning of others. Danielle says multiplying by $\cos\theta$ is the same as dividing by $\csc\theta$. Is she correct? Explain your reasoning.

ACTIVITY 33 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 33-1

1. In which quadrant are sine, cosine, and tangent all positive?
A. I **B.** II **C.** III **D.** IV

2. In which quadrant are both sine and cosine negative?
A. I **B.** II **C.** III **D.** IV

3. Given that $\cos\theta = -\frac{11}{61}$ and that $\pi < \theta < \frac{3\pi}{2}$, what is the value of $\sin\theta$?
A. $-\frac{11}{60}$ **B.** $\frac{11}{60}$
C. $-\frac{60}{61}$ **D.** $\frac{60}{61}$

4. Given that $\cos\theta = \frac{15}{17}$ and that $\frac{3\pi}{2} < \theta < 2\pi$, find the value of $\sin\theta$ and $\tan\theta$.

5. Given that $\sin\theta = \frac{5}{13}$ and that $0 < \theta < \frac{\pi}{2}$, find the value of $\cos\theta$ and $\tan\theta$.

6. Given that $\sin\theta = -\frac{6\sqrt{2}}{12}$ and that $\frac{3\pi}{2} < \theta < 2\pi$, what is the value of $\cos\theta$?
A. $\frac{6\sqrt{2}}{12}$ **B.** $-\frac{6\sqrt{2}}{12}$
C. 1 **D.** -1

7. Given that $\sin\theta = \frac{2\sqrt{3}}{4}$ and that $\frac{\pi}{2} < \theta < \pi$, find the value of $\cos\theta$ and $\tan\theta$.

8. Given that $\cos\theta = -\frac{3\sqrt{2}}{6}$ and that $\pi < \theta < \frac{3\pi}{2}$, find the value of $\sin\theta$ and $\tan\theta$.

9. If the sine of an angle is positive and the cosine is negative, in which quadrant is the terminal side of the angle?
A. I **B.** II **C.** III **D.** IV

ACTIVITY 33 Continued

ACTIVITY PRACTICE

1. A
2. C
3. C
4. $-\frac{8}{17}, -\frac{8}{15}$
5. $\frac{12}{13}, \frac{5}{12}$
6. A
7. $-\frac{2}{4}, -\frac{2\sqrt{3}}{2}$
8. $-\frac{3\sqrt{2}}{6}, 1$
9. B

10. $\sin\theta$
11. B
12. $\tan\theta$, $\frac{\sin\theta}{\cos\theta}$
13. C
14. A
15. D
16. $\sin^2\theta + \cos^2\theta = 1$
17. $1 + \cot^2\theta = \csc^2\theta$
18. 1
19. It can be true. If $\alpha = \theta$, the equation is true. If the reference angle for α is equal to the reference angle for θ, the equation is also true. Otherwise, it is false.

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

Lesson 33-2

10. Simplify $\frac{1}{\csc\theta}$.

11. Which is $\frac{1}{\cos^2\theta}$ simplified?
A. $\sin^2\theta$ **B.** $\sec^2\theta$
C. $\csc^2\theta$ **D.** $\cot^2\theta$

12. Write $\frac{1}{\cot\theta}$ in two other ways.

13. Which expression(s) equal $\sin\theta$?
I. $\frac{1}{\csc\theta}$ III. $\frac{1}{\sec\theta}$
II. $\csc\theta$ IV. $\tan\theta\cos\theta$
A. I only
B. III only
C. I and IV
D. II and IV

14. Which expression(s) are not equal to $\tan^2\theta$?
I. $\sec^2\theta + 1$ III. $\frac{\sin^2\theta}{\cos^2\theta}$
II. $\sec^2\theta - 1$ IV. $\frac{1}{\cot^2\theta}$
A. I
B. II
C. III
D. I and IV

15. Which is the product of $\tan^2\theta + 1 = \sec^2\theta$ and $\cos^2\theta$?
A. $\csc^2\theta + \cos^2\theta = \cos^2\theta\sec^2\theta$
B. $\sec^2\theta + \cos^2\theta = \cot^2\theta$
C. $\sec^2\theta + \cos^2\theta = 1$
D. $\sin^2\theta + \cos^2\theta = 1$

16. Multiply $1 + \cot^2\theta = \csc^2\theta$ by $\sin^2\theta$ to find another form of the trigonometric identity.

17. Multiply $\sin^2\theta + \cos^2\theta = 1$ by $\csc^2\theta$ to find another form of the trigonometric identity.

18. What is the product of $\tan\theta$ and $\cot\theta$?

MATHEMATICAL PRACTICES
Make Sense of Problems and Persevere in Solving Them

19. Is $\sin^2\theta + \cos^2\alpha = 1$ a true equation? Explain.